such displacements, is arrived at in the paper and illustrated.

(*e*) A simple criterion is also given for detecting errors in star measures due to curvature of a plate, which is copied on to another without being completely in contact with it, under oblique illumination.

On the Errors of Star Photographs due to optical Distortion of the Object-glass with which the Photograph is taken. Second paper. By H. H. Turner, M.A., F.R.S., Savilian Professor.

1. In a previous paper on this subject I gave the comparison of two Astrographic Catalogue plates taken at Oxford with corresponding portions of a large plate taken at Arequipa, Peru, with a photographic doublet. In each case the coordinates of some 250 stars on the two plates were compared, and the algebraical relations between them were found to be expressible with sensible accuracy by the formulæ

$$x' = c + ax + by + \mathrm{A}x^2 + \mathrm{B}xy + \mathrm{C}y^2$$
$$y' = f + dx + ey + \mathrm{D}x^2 + \mathrm{E}xy + \mathrm{F}y^2$$

The linear terms $c + ax + by$ and $f + dx + ey$ were not further considered; but it was pointed out that the values of the coefficients A B C D E F were inconsistent with the idea that these terms were due to optical distortion of the object-glass, if the corresponding displacement of a star image was along the radius from some centre, and a function of the distance from the centre. It was shown to be probable that these terms represented errors of copying the Arequipa plate under oblique illumination; and it was inferred that the optical distortion over a field of at least $4° \times 4°$ was small.

2. In the present paper additional evidence is given of the existence of errors of copying. Measures of a third region still further from the centre of the Arequipa plate are examined in the terms of the second order, which still show no evidence of sensible optical distortion. Finally, the linear terms $c + ax + by$ and $f + dx + ey$, hitherto neglected, are considered; again without finding evidence of optical distortion.

Errors of Copying.

3. In the previous paper no direct evidence was given that the errors were due to copying; but this was shown to be a probable explanation of the discrepancies. It was found that the glass of the Arequipa positive was sensibly curved; and it was supposed that in copying a portion of it on another plate the

films were not in complete contact owing to this curvature. Hence, if the illumination used for copying were slightly oblique, the star images in different parts of the plate would be relatively displaced. Thus, in fig. 1, if A S B be one (curved) film, A M N B the other, in contact with the former at A and B, but not at S; then if the illumination be in the direction S M, the star image S appears at M; if in the direction S N, at N; while the stars at A and B are the same in both cases.

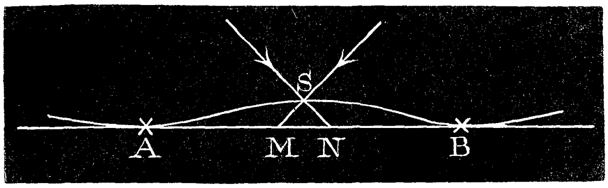

FIG. 1

4. To test this hypothesis a portion of the Arequipa plate was copied twice on the same plate under illuminations differing considerably in direction, and the distances between the pairs of images measured. The plate was held in the first instance so that light from a gas jet fell at an angle of $+45°$ with the normal, in the plane through the axis of x; then after a small displacement it was again exposed, with the light making an angle of $-45°$. It will be remarked that here we are not concerned with the performance of the Bruce doublet in any way. The stars are merely arbitrary points on the plate copied twice over, and the only differences allowable are of the form

$$by + c \text{ in } x \text{ and } -bx + f \text{ in } y$$

representing an arbitrary displacement and rotation in moving the plate slightly, so that the second exposure should not fall on the first. The mean differences of groups in x and y corrected by expressions of the above form were as below, in the notation of the previous paper, the unit being ·0001 of a *réseau* interval, or $0''·03$.

In x				In y			
−5	+3	+17	−9	−21	−14	−8	−7
−14	+7	+36	+7	−13	−9	−6	−2
−13	+7	+24	+6	−19	−9	0	+8
−10	−7	+17	+8	−18	−5	−1	+12

5. In the same way as in the former paper we find that algebraical expressions representing these are:

$$\text{for } x, \text{ const.} + ·00013x + ·00001y - ·000001(19x^2 + 2xy + 6y^2)$$
$$\text{for } y, \text{ const.} + ·00010x - ·00004y - ·000001(1x^2 + 5xy + 2y^2)$$

These expressions reduce the above differences to the following :

In x				In y			
+12	−5	+4	−14	−3	0	+2	+1
−1	−5	+16	−4	+3	+2	−3	−2
+2	−5	+4	−7	−1	0	+1	+2
+13	−11	+3	+1	+2	+5	−3	+2

6. If the differences are due to errors of the kind suggested, then both the algebraical expressions and the residuals in x and y should be in a constant ratio. Looking first at the residuals, take the five largest residuals in x, and write under them the corresponding residuals in y.

$$x \quad +16+12-14+13-11$$
$$y \quad -3-3+1+2+5$$

The ratio of the y residuals to the x residuals is thus small, its mean value being $-10/66$; and there is a reason for this, the obliquity of illumination being changed principally in the direction of the x axis, though the position of the plate being only roughly estimated, there was probably a slight change of direction in the other coordinate also. Thus corresponding to the term $-19x^2$ for x, we should expect a term $+3x^2$ in y, whereas we find $-1x^2$; but this need not trouble us, for coefficients smaller than 10 in these terms may be regarded as largely accidental.

In considering the terms of the first order, it must be remembered that we have an unknown rotation of one exposure with reference to the other ; thus, instead of

$$13x+1y \text{ and } 10x-4y$$

we should write

$$13x+(1+b)y \text{ and } (10-b)x-4y.$$

Putting $b=13$ we get

$$13x+14y \text{ and } -3x-4y,$$

which are approximately in the ratio of 66 to -10. So that there is no theoretical difficulty in getting this accordance between these terms and the residuals. But this is treating too seriously terms of the magnitude of $-3x-4y$, which may be in error by several units. It is probably better to put $b=5$ and get

$$+13x+6y \text{ for } x, \text{ and } 5x-4y \text{ for } y,$$

of which we can regard as largely accidental the smaller terms, *i.e.* all except the $+13x$.

Hence, the main part of the discrepancy between exposures is represented by

$$\text{Const.} + \cdot 000130\, x - \cdot 000019\, x^2 \text{ in } x$$

and

$$\text{Zero} \qquad \text{in } y$$

neglecting small terms as largely accidental.

This can be put into the form

$$\text{Const.} - \cdot 000019\, (x - 3\cdot 4)^2$$

and thus $x = 3\cdot 4$ is the point, either of closest contact between the two films in copying, or of greatest separation. Since it is more likely that the edges of the superimposed plate were in contact with the film to be copied, with a gap in the middle, rather than that the contact was in the middle with separation at the edges, $x = 3\cdot 4$ is probably the point of greatest separation. But this is not material.

7. What *is* rather important is the numerical smallness of the quantity. The obliquity of illumination was purposely exaggerated, and we should have expected at least five times the sort of quantities which were to be explained in the last paper, such as

$$+ 11x^2 - 23xy - 3y^2$$

(see p. 451); whereas, the quantity obtained is only about the same size. This is capable of explanation in more than one way, thus :—

(*a*) The plates may have been pressed closer together on the present occasion.

(*b*) The errors dealt with in the last paper may not be wholly due to the copying at Oxford, but partly to the process of copying the original negative at Arequipa.

8. In any case the existence of errors of this kind is fairly well established, and we may now return to the main object of the investigation, the possible existence of optical distortion. The next step was to take a copy of a portion of the Arequipa plate still further from the centre: to compare it with the corresponding Oxford plate as before: to express the residuals, after clearing away linear terms, by expressions of the second order, and to see whether at this greater distance from the centre these were becoming larger than could be explained by errors of copying.

9. The centres of the two plates previously considered occupied the following positions on the Arequipa plate,

No.	x	y	r
1144	+12	−7	14
835	−9	−6	11

N N

The new plate taken was with centre,

207 −25 +6 26

To the Oxford measures were applied the corrections,

$$+\cdot 220 - \cdot 010x - \cdot 004y \text{ to } x$$
$$0 \quad + \cdot 004x - \cdot 010y \text{ to } y$$

and it was then compared with the corresponding portion of the Arequipa plate. The mean differences were as below, the unit being 0″·03.

In x				In y			
+22	+165	+289	+368	+274	+177	+70	+9
+39	+151	+304	+368	+183	+116	+23	−40
−14	+116	+262	+311	+109	+59	−5	−35
−10	+96	+209	+282	+37	+11	−13	−37

10. When corrected by the linear expressions

$$-\cdot 0185 - \cdot 00169\, x - \cdot 00042y \text{ in } x,$$

and

$$-\cdot 0059 + \cdot 00092\, x - \cdot 00064y \text{ in } y,$$

these become

In x				In y			
−36	−3	+3	−28	+59	+22	−21	−22
+8	+10	+45	−1	+10	+3	−26	−29
−15	+5	+33	−28	−20	−10	−10	+20
+16	+12	+7	−30	−50	−16	+24	+60

11. Calculating for these corrections of the second order as in the previous paper, we find

$$(+32x^2 - 10xy + 16y^2) \times \cdot 000001$$
$$(-\ 9x^2 + 54xy - 17y^2) \times \cdot 000001.$$

Now the centre being at (−25, +6), or the coordinates of the centre of the Arequipa plate on this plate being (+25, −6), the theoretical corrections for difference of normals

$$\mu^2\,(Xx^2 + Yxy) \text{ and } \mu^2\,(Xxy + Yy^2)$$

(where μ is the circular measure of the *réseau* interval or $\mu^2 = \cdot 0000021$) are respectively

$$+53x^2 - 13xy \text{ and } +53xy - 13y^2,$$

both multiplied by ·000001. Hence the outstanding residuals require correction by

$$-21x^2+3xy+16y^2 \text{ for } x$$

and

$$-\ 9x^2+xy-4y^2 \text{ for } y.$$

12. Calling these $Ax^2+Bxy+Cy^2$ and $Dx^2+Exy+Fy^2$, then if they were due to optical distortion, according to the criterion of the last paper, A, E, C should be of the same sign and in descending order numerically; also F, B, D should be of the same sign and in descending order. This is far from being the case, A and C being conspicuously of opposite signs.

13. On the other hand, if these corrections have their origin in errors of copying we should have

$$\frac{A}{D}=\frac{B}{E}=\frac{C}{F}.$$

These relations are not fulfilled as the numbers stand, but a small "error of normal" could be assigned which would make them fit, *e.g.* if X and Y above instead of being (−25, +6) are really (−21, +12), we should get for the theoretical corrections for difference of normals

$$+44x^2-25xy \text{ and } +44xy-25y^2,$$

and thus

$$Ax^2+Bxy+Cy^2=-12x^2+15xy+16y^2$$
$$Dx^2+Exy+Fy^2=-9x^2+10xy+8y^2$$

14. Of course this is introducing two new arbitrary constants, and we must therefore not attach too much importance to the result; but it is to be remarked that not even by this device can we make the expressions such as would arise from optical distortion, for we cannot make A of the same sign as C and numerically greater without assuming an extravagant error of normal, viz. 18 *réseau* intervals at least.

15. Hence it would appear from the study of the terms of the second order that not even at this increased distance from the centre of the Arequipa plate is there evidence of sensible optical distortion; such terms of this order as affect the residuals being due to errors of tilt or copying.

16. But, as above stated, my attention was recalled by a remark of Mr. Dyson to the linear terms which may theoretically be larger than those of the second order, with a certain law of optical distortion, and the study of them gave interesting results. In explanation of the fact that I have hitherto neglected them, I may remark that in commencing this rather new investigation my attention was naturally attracted by the newest thing which I encountered. Had no terms of the second order presented themselves, the linear terms would naturally have been carefully

examined. But terms of the second order (not due to a difference of centres) in the comparison of two plates were a new feature. In comparing our astrographic plates the relations are sensibly linear (except for any difference of centring), and in looking for optical distortion on these large plates taken with a doublet, which even near the centre could not be expressed in terms of our plates by linear relations, I felt that at any rate these terms of the second order must be explained, and they seemed to afford the quickest route to the solution of the problem. The time spent in studying them was in no way lost, for I doubt whether I should have come upon the errors of copying by study of the linear terms.

The Linear Terms.

17. Before examining the individual plates the nature of the problem may be briefly stated. Each Oxford plate has been compared with the theoretical sky in the course of the regular work at Oxford, and is therefore as good as a piece of sky, except for accidental errors. Each copy of a portion of the Arequipa plate has been compared with an Oxford plate and thus indirectly with the sky. The formula for reduction of a copy to the sky will include

(*a*) The effect of refraction for the Arequipa plate ;
(*b*) The effect of aberration for the Arequipa plate ;
(*c*) The scale value of the Arequipa plate ;
(*d*) An arbitrary orientation and error of centring of the particular copy ;
(*e*) The correction for scale of the particular copy due to its distance from the centre of the large plate ;
(*f*) the effect of optical distortion.

18. As regards (*b*) the effect of aberration to the first order is a simple change of scale, and hence it may be considered as included in (*c*).

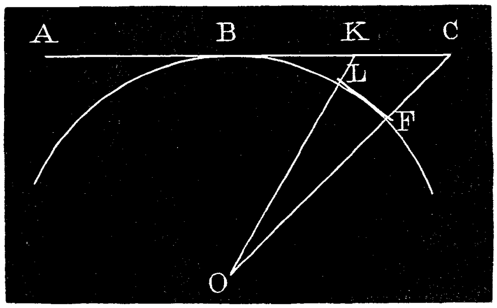

FIG. 2.

The correction (*e*) requires, perhaps, a little more explanation. Let A B C (fig. 2) be the large Arequipa plate. Then a portion, K C, far from the centre, is compared with an Oxford plate, which may be regarded as touching the celestial sphere at L F ; and

since K C is greater than L F, the scale of the Arequipa plate will in this region be relatively larger than at the centre B. The theoretical correction will be given later.

19. Coming to (f) the optical distortion, if there be displacements $\phi(x, y)$ and $\psi(x, y)$ in the coordinates x and y due to optical distortion, then in the neighbourhood of the point (ξ, η) these have the values

$$\phi(\xi_1\eta) + x\phi + y\phi_\eta + \tfrac{1}{2}(x^2\phi_{\xi\xi} + 2xy\phi_{\xi\eta} + y^2\phi_{\eta\eta}) + \&c.,$$

with a similar expression for ψ, and the linear terms are thus

$$x\phi_\xi + y\phi_\eta \text{ and } x\psi_\xi + \eta\psi_\eta.$$

As in the previous paper (§ 35), if the displacement be along a radius from some centre, and vary as r^{n+1}, so that the displacements in x and y are $\mathrm{P}r^n x = \phi(x, y)$ and $\mathrm{P}r^n y = \psi(x, y)$, then

$$\phi_\xi = \mathrm{P}(\rho^n + n\rho^{n-2}\xi^2) \qquad \phi_\eta = \mathrm{P}n\rho^{n-2}\xi\eta$$
$$\psi_\xi = \mathrm{P}n\rho^{n-2}\xi\eta \qquad \psi_\eta = \mathrm{P}(\rho^n + n\rho^{n-2}\eta^2).$$

As regards the magnitude of these compared with terms of the second order, we may take as an instance

$$\phi_{\xi\xi} = n\mathrm{P}\rho^{n-4}\xi\{3\rho^2 + (n-2)\xi^2\}.$$

Hence the ratio of $\frac{1}{2}\phi_{\xi\xi}x^2$ to $\phi_\xi \,.\, x$ is

$$\tfrac{1}{2}n\xi x\{3\rho^2 + (n-2)\xi^2\} \text{ to } \rho^2\{\rho^2 + n\xi^2\}.$$

The limiting cases are $\xi = \rho$ and $\xi = 0$. When $\xi = \rho$ the ratio is

$$\tfrac{1}{2}nx \text{ to } \rho.$$

When $\xi = 0$ the ratio is zero. Thus if n be large the terms of the second order are more important than the linear terms; but if n be small this is not so.

20. We proceed to the consideration of the three regions measured on the Arequipa plate, and compared with Oxford plates 1144, 835, and 207. In the first instance the Oxford measures, say x and y, were corrected to x' and y', values near those of the Arequipa copies, to help in readily identifying the proper stars. The values of $x' - x$ and $y' - y$ are given in §§ 26 and 45 of the former paper and § 9 of the present one, and are as below :

Plate.	$x' - x$.	$y' - y$.
1144	$-\cdot 010x + \cdot 002y$	$-\cdot 002x - \cdot 010y$
835	$-\cdot 010x + \cdot 003y$	$-\cdot 003x - \cdot 010y$
207	$-\cdot 010x - \cdot 004y$	$+\cdot 004x - \cdot 010y$

21. We then found that if (X, Y) denote measures on a copy of the Arequipa plate, and (x', y') denote the Oxford measures

corrected as above, the values of $X-x'$ and $Y-y'$ were as follows, omitting a constant :

Plate.	$X-x'$.	$Y-y'$.
1144	$+\cdot00017X+\cdot00000Y$	$-\cdot00051X+\cdot00014Y$
835	$-\cdot00014X-\cdot00019Y$	$+\cdot00008X-\cdot00062Y$
207	$+\cdot00169X+\cdot00042Y$	$-\cdot00092X+\cdot00064Y$

22. Finally, in the course of the regular work at Oxford the following expressions were found for reduction of (x, y) to (ξ, η), the standard coordinates for stars in the theoretical sky :

Plate.	$x-\xi$.	$y-\eta$.
1144	$+\cdot00750\xi+\cdot01435\eta$	$-\cdot01418\xi+\cdot00756\eta$
835	$+\cdot00835\xi+\cdot00792\eta$	$-\cdot00769\xi+\cdot00826\eta$
207	$+\cdot00781\xi+\cdot00117\eta$	$-\cdot00150\xi+\cdot00791\eta$

23. We thus obtain for $X-\xi$ and $Y-\eta$ the following values:

Plate.	$X-\xi$.	$Y-\eta$.
1144	$-\cdot00244\xi+\cdot01624\eta$	$-\cdot01658\xi-\cdot00242\eta$
835	$-\cdot00189\xi+\cdot01068\eta$	$-\cdot01055\xi-\cdot00246\eta$
207	$-\cdot00058\xi-\cdot00244\eta$	$+\cdot00162\xi-\cdot00153\eta$

There is thus a well-marked difference in scale value between the copies, but this is in great measure due to the cause specified in (*e*) of § 17, and we must now estimate this effect numerically.

24. The relations between coordinates $(x_1 y_1)$ on one plate and $(x_2 y_2)$ of the same star on another plate, whose centre is at (XY) on the first, are approximately

$$x_2 = \frac{x_1 - X}{1 + Xx_1 + Yy_1} \qquad y_2 = \frac{y_1 - Y}{1 + Xx_1 + Yy_1}$$

or

$$x_2(1 + Xx_1 + Yy_1) = x_1 - X,$$

or substituting on the left

$$x_1 = x_2 + X \text{ and } y_1 = y_2 + Y$$

$$x_2(1 + X^2 + Y^2) + Xx_2^2 + Yx_2y_2 = x_1 - X;$$

and similarly

$$y_2(1 + X^2 + Y^2) + Xx_2y_2 + Yy_2^2 = y_1 - Y.$$

Thus the change of scale is represented by the factor $(1+X^2+Y^2)$ and depends simply on the distance between centres, as it should. If X and Y are expressed in *réseau* intervals, we must multiply X^2+Y^2 by $\cdot0000021$, and the products for the three plates are (see § 9).

Plate.	Product.
1144	·00041
835	·00024
207	·00138

25. In applying these corrections we may at the same time rotate the axes so as to reduce the coefficients of y and x in x and y respectively. If these were exactly equal, and of opposite sign, we could make them both zero by a simple rotation of axis, writing

$$x_2 \text{ for } (1-\tfrac{1}{2}\theta^2)x_1 - \theta y_1$$

and

$$y_2 \text{ for } +\theta x_1 + (1-\tfrac{1}{2}\theta^2)y_1.$$

But the equality is only approximate, and we choose for θ the numerical mean of the two values, as below :

Plate.	$+\theta$.	$\frac{1}{2}\theta^2$.
1144	$-·01641$	·00013
835	$-·01061$	·00005
207	$+·00203$	·00000

26. The corrected values of $X-\xi$ and $Y-\eta$ thus become

Plate.	$X-\xi$.	$Y-\eta$.
1144	$-·00272\xi-·00017\eta$	$-·00017\xi-·00271\eta$
835	$-·00228\xi+·00007\eta$	$+·00006\xi-·00265\eta$
207	$-·00196\xi-·00041\eta$	$-·00041\xi-·00291\eta$

27. The correction for differential refraction on the Arequipa plate must now be calculated (the refraction and aberration for the Oxford plates are included in the formulæ, reducing them to the theoretical sky). Particulars are given in § 50 of the preceding paper for calculating X_0, Y_0, the coordinates of the zenith on the Arequipa plate, which are thus found to be $X_0=-0·44$ and $Y_0=-0·99$; and the corrections for differential refraction to the first order, viz.

$$+\mu(1+X_0^2)X+\mu X_0Y_0 . Y \text{ in } x$$

and

$$+\mu X_0Y_0 . X+\mu(1+Y_0^2)Y \text{ in } y$$

(where μ is the coefficient of refraction and may be taken as 57″, ·00028 in circular measure) thus become

$$+·00033X+·00011Y \text{ and } +·00011X-·00054Y.$$

The effect of refraction is to decrease X and Y by these expressions : and hence the measured X and Y are too small to this extent. To compare X and Y with ξ and η as if there were no refraction we must therefore add the above expressions to X and Y, making $X-\xi$ and $Y-\eta$ larger in consequence, as below.

Plate.	$X-\xi$.	$Y-\eta$.
1144	$-·00239\xi-·00006\eta$	$-·00006\xi-·00217\eta$
835	$-·00195\xi+·00018\eta$	$+·00017\xi-·00211\eta$
207	$-·00163\xi-·00030\eta$	$-·00030\xi-·00237\eta$

28. If there were no optical distortion, then denoting the value of

$$X-\xi \text{ by } a_1\xi+b_1\eta,\ a_2\xi+b_2\eta, \ \&c.$$

and of

$$Y-\eta \text{ by } +b_1\xi+e_1\eta,\ +b_2\xi+e_2\eta, \ \&c.$$

we should expect $a_1=a_2=a_3=e_1=e_2=e_3$, and $b_1=b_2=b_3=0$. The departure from this state of things is quite sensible, and it remains to examine whether the departure can be attributed to optical distortion.

29. The values of the coefficients a, b, and e have been given in §19; but the symbols there employed have since been used in rather different senses. Let us put (u, v) for the coordinates of the centre of a plate or region referred to the optical centre, and let $r^2=u^2+v^2$. Then if the distortion is Pr^n, we have the following values for the coefficients a, b, and e:—

$$a=Pr^{n-2}(r^2+nu^2),\quad b=Pnr^{n-2}\,.\,u\,v,\quad e=Pnr^{n-2}(r^2+nv^2).$$

Thus

$$\frac{a-e}{b}=\frac{u^2-v^2}{uv},$$

a relation which is independent of n, and affords a simple test of the existence of this kind of optical distortion.

30. Assuming that the optical centre is at the centre of the Arequipa plate, the values of u and v for the three plates are given in § 9 under the head x and y: and the following tabular statement shows how far the above relation is fulfilled.

Plate.	$a-e$.	b.	$\frac{a-e}{b}$	u^2-v^2	uv	$\frac{u^2-v^2}{uv}$
1144	−·00022	−·00006	+3·7	+ 95	− 98	−1·0
835	+·00016	+·00017	+1·0	+ 45	+ 54	+0·8
207	+·00074	−·00030	−2·5	+589	−150	−3·9

There is a considerable discrepancy in the case of Plate 1144; but the coefficient b is small, and may be largely accidental. It may be remembered that this plate is a good deal affected by errors of copying.

31. Since $b/uv=Pnr^{n-2}$, a comparison of the three values of b/uv should be some guide to the value of n. Take the second and third plates, which seem fairly accordant. The values of b/uv are in the ratio 31 to 20, which is $(r_2/r_3)^{n-2}$. But $r_2/r_3=11/26$. This suggests that $n-2$ is negative, the actual value given by this relation being $n-2=-0{\cdot}5$ or $n=1{\cdot}5$. If for the same plates we took $(a-e)/(u^2-v^2)$, which should also give Pnr^{n-2}, the ratio is 36 to 13, giving $n-2=-1{\cdot}2$ or $n=0{\cdot}8$. Hence the value $n=1$, implying a law of displacement $=Pr^2$, with displacements Prx and Pry in x and y, does not seem unlikely.

32. Calculating on this supposition

$$\frac{a}{P} = r + \frac{u^2}{r}, \quad \frac{b}{P} = \frac{uv}{r}, \quad \frac{e}{P} = r + \frac{v^2}{r}$$

for the three plates, we get the following values :

Plate.	a	b	$\frac{e}{P}$
1144	+24	−6	+17
835	+18	−5	+14
207	+51	−6	+27

33. It remains to assign the value of P. From consideration of this table in conjunction with that of § 27, we may put P = + ·00002, and then subtract these theoretical values of a, b, and e from those of § 27. If this law of distortion is satisfactory, we ought then to get the values of a and e all equal, and the values of b all zero. The old values and the new are collected in the following table. To show more clearly the differences of a and e, the mean value −210 has been subtracted from all the old values, and the mean value −261 from all the new values ; the unit is as before, ·00001.

Plate.	Old Values.			New Values.		
	a	e	b	a	e	b
1144	−29	− 7	− 6	−26	+10	+ 6
835	+15	− 1	+18	+30	+22	+ 8
207	+47	−27	−30	− 4	−30	−18

34. We have not therefore improved things very much. The mean numerical value of b is reduced from 18 to 11, but that of a and e only alters from 21 to 20. Nor can we get any much better result by taking another value of n : it can be shown that there is an inherent difficulty in satisfying the conditions, independent of n. For write the old values of a and e which it is desired to bring into accordance in the order of magnitude, viz. +47 +15 −1 −7 −27 −29, then the corresponding expressions $Pr^{n-2}(r^2+nu^2)$ &c. (see § 29) should fall into approximately the same order. Picking these out, and substituting the values of u, v, and r, from § 9 we get the following series, which should be in order of magnitude.

$$26^n(1+\cdot 8n),\ 11^n(1+\cdot 8n),\ 11^n(1+\cdot 3n),\ 14^n(1+\cdot 3n),\ 26^n(1+1n),\ 14^n(1+\cdot 8n)$$

When n is small we may take logarithms to base e, and neglecting powers of n above the first we can divide out by n. Thus the first term is n (log. 26 + ·8), and dividing by n the following numbers should be in order :—

1·8	0·9	0·4	0·7	1·1	1·1

which is not even approximately the case. Large values of n are excluded by other considerations.

35. Hence it is no easier to detect optical distortion in the linear terms than in those of the second order, and we must regard the discrepancies as due to accidental causes—perhaps errors of copying—in the same way as the second order terms.

36. Thus we may feel some confidence that the optical distortion over a considerable field is small; and that accurate positions of stars may be obtained from these large plates if proper precautions be taken. I do not think, however, that the present line of examination is the best way of arriving at precise information about a small optical distortion; the method of trails, mentioned by Captain Hills, R.E., at the April meeting of the Society, seems better.

In another paper I have examined the necessary formulæ of reduction, and find them very simple; and I think the method will prove to be a very easy and direct way of measuring the optical distortion of a lens. The foregoing investigations will, however, serve to show how two plates may be compared when we have no independent information as to the optical distortion.

Conclusions of the Present Paper.

1. The errors of copying due to curvature of plate and oblique illumination, suggested in the last paper, were reproduced by direct experiment. The numerical value found was smaller than was expected, though this is capable of explanation. (§§ 1–7.)

2. A third region, including stars up to 4° from the centre of the Arequipa plate, was compared with an Oxford plate, and the terms of the second order gave no evidence of optical distortion. (§§ 8–15.)

3. The terms of the first order for all three plates dealt with in this paper and the last were examined, and gave no evidence of optical distortion. (§§ 16–35.)

4. Hence it seems possible to obtain good results over a region of (say) $5° \times 5°$ with a photographic doublet.

5. Further experiments are, however, desirable by the method of star-trails, as indicated elsewhere.

On the Curvature of Star-trails on a Photographic Plate as a Means of Investigating Optical Distortion. By H. H. Turner, M.A., F.R.S., Savilian Professor.

1. In the discussion on my paper on optical distortion, read at the April meeting of the Society, Captain Hills, R.E., mentioned a simple method of investigating the distortion on a wide-angle photograph, which he had used in practice—viz. to take a series of star-trails on the plate and to compare their curvature with